A denotational semantics for a Lewis-style modal system close to S1

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Abstract

While possible worlds semantics provides a natural framework for normal modal logics, there is no such intuitive semantics for modal system S1 designed by C. I. Lewis as a logic of strict implication. In this paper, we interpret strict equivalence $\Box(\varphi \rightarrow \psi) \wedge \Box(\psi \rightarrow \varphi)$ as propositional identity $\varphi \equiv \psi$ (read: "φ and ψ denote the same proposition") and extend S1 by an inference rule which is a natural generalization of the rule of Substitutions of Proved Strict Equivalents. The resulting modal system is only slightly stronger than S1 and satisfies the principles of non-Fregean logic. This enables us to develop an intuitive, non-Fregean denotational semantics for which the system is sound and complete.

Keywords: modal logic, strict implication, strict equivalence, propositional identity, non-Fregean logic, denotational semantics

1 Introduction

Discontented with the notion of material implication found in the *Principia Mathematica*, C. I. Lewis introduced axiomatizations of *strict* implication [8, 9]. These systems are known as the non-normal modal logics S1 to S3. In [9] also appeared deductively equivalent systems of the modal logics S4 and S5 which, however,

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are not accepted by Lewis as systems of strict implication. Possible worlds semantics for modal logics was introduced years later by S. Kripke and has become the standard semantics. Inherent in this semantics are rather strong modal inference principles such as the Necessitation Rule. Some non-normal modal logics which do not validate the Necessitation Rule, such as the systems S2 and S3, can be described within the possible worlds framework if one admits the concept of a *non-normal world*. However, there seems to be no natural possible worlds semantics for S1. The only known semantics for S1, developed by Cresswell [4, 5], is unintuitive.[1]

An approach to modality that avoids the possible worlds framework and instead relies on the principles of non-Fregean logic (see, e.g., [1]) is studied by R. Suszko [14]. The specific syntactical ingredient of a non-Fregean logic is an identity connective $\equiv$. A formula $\varphi \equiv \psi$ can be read as "φ and ψ denote the same proposition" or "φ and ψ have the same meaning (Bedeutung)". Suszko introduces necessity by means of the identity connective: $\Box\varphi \equiv (\varphi \equiv \top)$ is an axiom, where the tautological sentence $\top$ denotes "the necessary proposition". Proceeding on these assumptions, Suszko defines two theories W_T and W_H which have certain topological Boolean algebras as models and correspond to the modal logics S4 and S5, respectively.[2] Axiomatic systems which define modality by means of an identity connective are also studied by some other authors, e.g. Cresswell [2, 3] and Martens [13]; see also the historical note at the end of [14].

A different non-Fregean approach to modal logic is developed by Lewitzka [11]. It extends the semantical principles of non-Fregean logic by the assumption that necessity is given as a subset of the propositional universe of a model — a viewpoint which is also the underlying idea of the earlier work [10]. Lewitzka axiomatizes (in an extended non-Fregean language) some well-known normal modal systems as first-order theories of propositions and shows that these modal systems can be captured (in a precise sense) by the proposed non-Fregean semantics.

In the present paper, we adopt from [10, 11] the ontological view on a modality as a set of propositions. But instead of a language with a freely interpretable identity connective we start here with the pure modal propositional logic and *define* a new connective via strict implication: $\varphi \equiv \psi := \Box(\varphi \to \psi) \wedge \Box(\psi \to \varphi)$. Formula $\varphi \equiv \psi$ expresses *strict equivalence* between φ and ψ. If some theory in this pure modal language contains the following Substitution Principle (SP) $(\psi \equiv \psi') \to (\varphi[x := \psi]) \equiv \varphi[x := \psi'])$, then $\equiv$ is in fact the identity connective

[1] This is pointed out on p. 202 in [6].

[2] A famous result due to McKinsey and Tarski is used here.

of non-Fregean logic and we can hope to find a natural semantics for that theory. It turns out that under our assumptions the axiom scheme SP can be seen as a natural generalization of an inference rule of system S1. We are able to present a non-Fregean denotational semantics for which the system S1+SP is sound and complete.

2 The deductive system

The set Fm of formulas of modal propositional logic is defined in the usual way over a set $V = \{x_0, x_1, x_2, ...\}$ of propositional variables, logical connectives $\neg, \rightarrow$ and the modal operator $\Box$ for necessity. If x is a variable and φ, ψ are formulas, then we write $\varphi[x := \psi]$ for the formula that we obtain by substituting ψ for x in φ. We use the following abbreviations:

- $\varphi \wedge \psi := \neg(\varphi \rightarrow \neg\psi)$
- $\varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$
- $\varphi \equiv \psi := \Box(\varphi \rightarrow \psi) \wedge \Box(\psi \rightarrow \varphi)$

The following axiomatization of Lewis system S1 is due to E. J. Lemmon [7], see also [6]. In contrast to Lewis original axiomatization found in [9], Lemmon's axiomatization is formulated as an extension of the calculus of (non-modal) propositional logic. All formulas of the following form are axioms:

(i) formulas which have the form of a propositional tautology

(ii) $\Box\varphi \rightarrow \varphi$

(iii) $(\Box(\varphi \rightarrow \psi) \wedge \Box(\psi \rightarrow \chi)) \rightarrow \Box(\varphi \rightarrow \chi)$.

The rules of the deductive system are

- Modus Ponens MP: "From φ and $\varphi \rightarrow \psi$ infer ψ."
- Axiom Necessitation AN: "If φ is an axiom, then infer $\Box\varphi$."
- Substitutions of Proved Strict Equivalents SPSE: "From $\psi \equiv \psi'$ infer any formula of the form $\varphi[x := \psi] \equiv \varphi[x := \psi']$."

We will see that S1 is sound with respect to the denotational semantics presented below. However, the system is apparently not complete. The problem is the rule SPSE which is too weak for non-Fregean semantics. We generalize SPSE to the following Substitution Rule SR:

"From $\chi \to (\psi \equiv \psi')$ infer $\chi \to (\varphi[x := \psi] \equiv \varphi[x := \psi'])$."

Then $(\psi \equiv \psi') \to (\varphi[x := \psi] \equiv \varphi[x := \psi'])$ is a theorem (choose $\chi = (\psi \equiv \psi')$ in rule SR). On the other hand, rule SPSE follows from the scheme $(\psi \equiv \psi') \to (\varphi[x := \psi] \equiv \varphi[x := \psi'])$ together with Modus Ponens. Thus, S1 extended with SR is deductively equivalent with the system that we obtain from S1 by replacing rule SPSE with the scheme of axioms

(iv) $(\psi \equiv \psi') \to (\varphi[x := \psi] \equiv \varphi[x := \psi'])$ (Substitution Principle SP)

This scheme is precisely the Substitution Principle SP of non-Fregean logic; see, e.g., Corollary 3.8 of [10] or Lemma 3.3 of [12]. In basic non-Fregean logic, this principle is established axiomatically in the form of the axioms (e)–(g) given in Definition 1.1 of [1]; see also the remark following Proposition 1.3 in [1]. In the following, we consider system S1 augmented with principle SP.

Definition 2.1 *For a set of formulas Φ we define $\Phi^{\vdash}$ as the smallest set that contains Φ together with all axioms given by the schemes (i) – (iv), and is closed under the rules MP and AN. If $\varphi \in \Phi^{\vdash}$, then we write $\Phi \vdash \varphi$ and say that φ is derivable from Φ.*

From our observations above it follows that $\vdash \varphi$ iff φ is a theorem of the system S1+SR. Since we prefer to work with the Substitution Principle SP instead of the rule SR, we will refer to that system as S1+SP. The next result follows by induction on a derivation.

Lemma 2.2 *(Deduction Theorem) If $\Phi \cup \{\varphi\} \vdash \psi$, then $\Phi \vdash \varphi \to \psi$.*

3 Denotational semantics

Definition 3.1 *A model is a structure $\mathcal{M} = (M, TRUE, NEC, f_\neg, f_\Box, f_\to, \leq)$, where M is a set of abstract entities, called propositions, $NEC \subseteq TRUE \subseteq M$, $\leq$ is a partial ordering on M, and $f_\neg : M \to M$, $f_\Box : M \to M$, $f_\to : M \times M \to M$ are functions satisfying the following:*

- $f_\neg(m) \in TRUE \Leftrightarrow m \notin TRUE$
- $f_\Box(m) \in TRUE \Leftrightarrow m \in NEC$
- $f_\rightarrow(m, m') \in TRUE \Leftrightarrow m \notin TRUE$ or $m' \in TRUE$
- $f_\rightarrow(m, m') \in NEC \Leftrightarrow m \leq m'$.

Definition 3.2 *Let $\mathcal{M}$ be a model. An assignment of propositions to formulas is a function $\gamma : V \rightarrow M$ which extends in the canonical way to a function $\gamma : Fm \rightarrow M$ (we refer to the extension again by γ). That is, $\gamma(\neg\varphi) = f_\neg(\gamma(\varphi))$, $\gamma(\Box\varphi) = f_\Box(\gamma(\varphi))$, and $\gamma(\varphi \rightarrow \psi) = f_\rightarrow(\gamma(\varphi), \gamma(\psi))$. An assignment $\gamma : V \rightarrow M$ is said to be good if $\gamma(\varphi) \in NEC$ for each axiom φ. If $\mathcal{M}$ is a model and γ is a good assignment of $\mathcal{M}$, then we call the tuple $(\mathcal{M}, \gamma)$ an interpretation. The satisfaction relation between interpretations and formulas is defined as follows: $(\mathcal{M}, \gamma) \vDash \varphi :\Leftrightarrow \gamma(\varphi) \in TRUE$. We say that the interpretation $(\mathcal{M}, \gamma)$ is a model of φ, or φ is true in $(\mathcal{M}, \gamma)$, if $(\mathcal{M}, \gamma) \vDash \varphi$. These definitions extend in the usual way to sets of formulas. For a set of formulas Φ we define $Mod(\Phi) = \{(\mathcal{M}, \gamma) \mid (\mathcal{M}, \gamma) \vDash \Phi\}$. The relation of logical consequence then is given as follows: $\Phi \Vdash \varphi :\Leftrightarrow Mod(\Phi) \subseteq Mod(\{\varphi\})$.*

The next result follows easily from the definitions. It says that the defined connective $\equiv$ has the intended semantics of an identity connective:

Theorem 3.3 *Let $(\mathcal{M}, \gamma)$ be an interpretation, and let φ, ψ be formulas. Then $(\mathcal{M}, \gamma) \vDash \varphi \equiv \psi \Leftrightarrow \gamma(\varphi) = \gamma(\psi)$.*

Let $\mathcal{M}$ be a model with universe M. With a given assignment $\gamma : V \rightarrow M$ one can associate a truth–value assignment $\beta_\gamma : V \rightarrow \{0, 1\}$ defined by $\beta_\gamma(x) = 1 :\Leftrightarrow \gamma(x) \in TRUE$. It follows that for all non-modal propositional formulas φ, we have that $\gamma(\varphi) \in TRUE \Leftrightarrow \beta_\gamma(\varphi) = 1$. Thus, $(\mathcal{M}, \gamma) \vDash \varphi$, for all tautologies φ of propositional logic. Since a (modal) formula φ, which has the form of a propositional tautology, is the result of certain replacements of variables in a tautology of propositional logic, it follows that also φ is true in $(\mathcal{M}, \gamma)$. From $NEC \subseteq TRUE$ it follows that all formulas of the form $\Box\varphi \rightarrow \varphi$ are satisfied. Formulas of the form $(\Box(\varphi \rightarrow \psi) \wedge \Box(\psi \rightarrow \chi)) \rightarrow \Box(\varphi \rightarrow \chi)$ are true since the relation $\leq$ on M is transitive. Let us show that SP is sound, i.e., $\Vdash (\psi \equiv \psi') \rightarrow (\varphi[x := \psi] \equiv \varphi[x := \psi'])$. Suppose $(\mathcal{M}, \gamma)$ is an interpretation such that $(\mathcal{M}, \gamma) \vDash (\psi \equiv \psi')$. By induction on φ, it follows that $\gamma(\varphi[x := \psi]) =$

$\gamma(\varphi[x := \psi'])$. That is, $(\mathcal{M}, \gamma) \vDash \varphi[x := \psi] \equiv \varphi[x := \psi']$. We conclude that all axioms are valid, i.e. true in all interpretations. Moreover, the notion of *good assignment* is well-defined: since $NEC \subseteq TRUE$, the requirement $\gamma(\varphi) \in NEC$ for axioms φ does not involve contradictions. It is clear that the rule MP is sound. The soundness of rule AN follows from the fact that we only consider *good* assignments.

Soundness of the calculus now follows by induction on derivations.

Theorem 3.4 (Soundness) *For any set* $\Phi \cup \{\varphi\} \subseteq Fm$*:* $\Phi \vdash \varphi \Rightarrow \Phi \Vdash \varphi$.

4 The Completeness Theorem

In order to prove completeness of S1+SP we follow the usual strategy. We call a set of formulas consistent if there is a formula which is not derivable from that set. A set which is not consistent is called inconsistent. A maximal consistent set of formulas is a consistent set such that every proper extension of it is inconsistent. From standard arguments it follows that each consistent set extends to a maximal consistent set. It remains to show that a maximal consistent set has a model. We will tacitly make use of the following well-known properties of a maximal consistent set Φ. These properties follow from propositional logic:

- $\varphi \in \Phi \Leftrightarrow \Phi \vdash \varphi$
- $\neg\varphi \in \Phi \Leftrightarrow \varphi \notin \Phi$
- $\varphi \rightarrow \psi \in \Phi \Leftrightarrow \varphi \notin \Phi$ or $\psi \in \Phi$.

Definition 4.1 *For a maximal consistent set* Φ *we define a relation* $\approx_\Phi$ *on* Fm *by* $\varphi \approx_\Phi \psi :\Leftrightarrow \Phi \vdash \varphi \equiv \psi$.

Lemma 4.2 *Let* Φ *be a maximal consistent set. The relation* $\approx_\Phi$ *is an equivalence relation on* Fm *with the following properties:*

- *If* $\varphi_1 \approx_\Phi \psi_1$ *and* $\varphi_2 \approx_\Phi \psi_2$*, then* $\neg\varphi_1 \approx_\Phi \neg\psi_1$, $\Box\varphi_1 \approx_\Phi \Box\psi_1$ *and* $\varphi_1 \rightarrow \varphi_2 \approx_\Phi \psi_1 \rightarrow \psi_2$.
- *If* $\varphi \approx_\Phi \psi$*, then* $\varphi \in \Phi \Leftrightarrow \psi \in \Phi$.
- *If* $\varphi \approx_\Phi \psi$*, then* $\Box\varphi \in \Phi \Leftrightarrow \Box\psi \in \Phi$.

Proof. Symmetry of $\approx_\Phi$ follows from propositional logic. Since $\varphi \to \varphi$ is an axiom, we get $\Box(\varphi \to \varphi) \in \Phi$ by rule AN. Thus, $\approx_\Phi$ is reflexive. Transitivity follows from the scheme of axioms (iii). We show the first item of the Lemma. Suppose $\varphi_1 \approx_\Phi \psi_1$ and $\varphi_2 \approx_\Phi \psi_2$. Let $x \neq y$ be variables such that x does not occur in ψ_2 and y does not occur in φ_1. Then by SP and MP: $(\varphi_1 \to \varphi_2) = (\varphi_1 \to y)[y := \varphi_2] \approx_\Phi (\varphi_1 \to y)[y := \psi_2] = (\varphi_1 \to \psi_2) = (x \to \psi_2)[x := \varphi_1] \approx_\Phi (x \to \psi_2)[x := \psi_1] = (\psi_1 \to \psi_2)$. Now we get the assertion by transitivity of $\approx_\Phi$. The remaining cases follow similarly. The second item of the Lemma follows from the schemes of axioms (iii) and (ii), and MP. Finally, the third item follows from the previous items of the Lemma. q.e.d.

Lemma 4.3 *A maximal consistent has a model.*

Proof. Let Φ be a maximal consistent set of formulas. For a formula φ, let $\overline{\varphi}$ be the equivalence class of φ modulo $\approx_\Phi$. The ingredients of our model are given by:

- $M := \{\overline{\varphi} \mid \varphi \in Fm\}$
- $TRUE := \{\overline{\varphi} \mid \varphi \in \Phi\}$
- $NEC := \{\overline{\varphi} \mid \Box\varphi \in \Phi\}$
- functions $f_\neg$, $f_\Box$, $f_\to$ defined by $f_\neg(\overline{\varphi}) := \overline{\neg\varphi}$, $f_\Box(\overline{\varphi}) := \overline{\Box\varphi}$, $f_\to(\overline{\varphi}, \overline{\psi}) := \overline{\varphi \to \psi}$, respectively
- a binary relation $\leq$ on M defined by $\overline{\varphi} \leq \overline{\psi} :\Leftrightarrow \Box(\varphi \to \psi) \in \Phi$.

By Lemma 4.2, these ingredients are well-defined. By propositional logic and AN, $\leq$ is reflexive. By scheme of axioms (iii), $\leq$ is transitive. Suppose $\overline{\varphi} \leq \overline{\psi}$ and $\overline{\psi} \leq \overline{\varphi}$. Then $\varphi \equiv \psi \in \Phi$ and therefore $\overline{\varphi} = \overline{\psi}$. Thus, $\leq$ is also antisymmetric. That is, $\leq$ is a partial order on M. It follows that $\mathcal{M} = (M, TRUE, NEC, f_\neg, f_\Box, f_\to, \leq)$ is a model. Finally, we consider the assignment $\gamma : V \to M$ defined by $\gamma(x) := \overline{x}$. It follows that $\gamma(\varphi) = \overline{\varphi}$, for any formula φ. By the rule AN, Φ contains the formula $\Box\varphi$ whenever φ is an axiom. So for any axiom φ we have $\gamma(\varphi) = \overline{\varphi} \in NEC$, by definition of NEC. That is, γ is a *good* assignment and thus $(\mathcal{M}, \gamma)$ is an interpretation. It follows that $\varphi \in \Phi \Leftrightarrow \overline{\varphi} \in TRUE \Leftrightarrow \gamma(\varphi) \in TRUE \Leftrightarrow (\mathcal{M}, \gamma) \vDash \varphi$. Thus, $(\mathcal{M}, \gamma) \vDash \Phi$. q.e.d.

Corollary 4.4 (Completeness Theorem) $\Phi \Vdash \varphi \Rightarrow \Phi \vdash \varphi$.

Proof. $\Phi \nvdash \varphi$ implies the consistency of $\Phi \cup \{\neg\varphi\}$. We extend this set to a maximal consistent set which, by Lemma 4.3, has a model. It follows that $\Phi \nVdash \varphi$. q.e.d.

A two-element model $\mathcal{M}$ has the following property: $(\mathcal{M}, \gamma) \vDash \varphi \equiv \psi$ iff φ and ψ have the same truth–value (i.e., both formulas are true or both formulas are false), for all formulas φ, ψ and all assignments $\gamma : V \to M$. In such an *extensional* or *Fregean* model, the denotation of a formula can be identified with its truth–value. Note that necessity collapses here with truth, and impossibility collapses with falsity. An *intensional* model is an interpretation $(\mathcal{M}, \gamma)$ such that for all formulas φ, ψ: $(\mathcal{M}, \gamma) \vDash \varphi \equiv \psi \Leftrightarrow \varphi = \psi$. In an intensional model, the denotation of a formula can be identified with its intension.[3] Whereas the construction of an intensional model for basic non-Fregean logic is straightforward, this becomes a complex task in the case of languages with propositional quantifiers (see, e.g., [12]). We observe here that the non-Fregean theory S1+SP has no intensional models since there are theorems of the form $\varphi \equiv \psi$ where $\varphi \neq \psi$. We believe, however, that there is a model $(\mathcal{M}, \gamma)$ with the following weaker property. For all formulas φ, ψ: $(\mathcal{M}, \gamma) \vDash \varphi \equiv \psi \Leftrightarrow\ \vdash \varphi \equiv \psi$. Such a model, if it exists, is in a sense *intensional modulo the theory* S1+SP. It satisfies only the minimal set of equations, i.e. only those required by the theory S1+SP. Constructions of such specific models remain to be further investigated. The same can be said about the application of our approach to other modal systems (of strict implication). Note that our approach relies on the T-axiom $\Box\varphi \to \varphi$, "necessity implies truth", which is assumed, e.g., in the proof of Lemma 4.2. In fact, this axiom is an ingredient of every Lewis modal system.

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